TWO CONJECTURES IN RAMSEY-TURÁN THEORY

JAEHOON KIM, YOUNJIN KIM AND HONG LIU

Abstract. Given graphs $H_1, \dots, H_k$, a graph G is $(H_1, \dots, H_k)$-free if there is a k-edge-colouring $\phi : E(G) \to [k]$ with no monochromatic copy of H_i with edges of colour i for each $i \in [k]$. Fix a function $f(n)$, the Ramsey-Turán function $\mathrm{RT}(n, H_1, \dots, H_k, f(n))$ is the maximum number of edges in an n-vertex $(H_1, \dots, H_k)$-free graph with independence number at most $f(n)$. We determine $\mathrm{RT}(n, K_3, K_s, \delta n)$ for $s \in \{3, 4, 5\}$ and sufficiently small δ, confirming a conjecture of Erdős and Sós from 1979. It is known that $\mathrm{RT}(n, K_8, f(n))$ has a phase transition at $f(n) = \Theta(\sqrt{n \log n})$. However, the value of $\mathrm{RT}(n, K_8, o(\sqrt{n \log n}))$ was not known. We determined this value by proving $\mathrm{RT}(n, K_8, o(\sqrt{n \log n})) = \frac{n^2}{4} + o(n^2)$, answering a question of Balogh, Hu and Simonovits. The proofs utilise, among others, dependent random choice and results from graph packings.

1. Introduction and results

Turán's theorem [26] states that among all n-vertex K_{s+1}-free graphs, the balanced complete s-partite graph, now so-called s-partite *Turán graph* $T_s(n)$, has the largest size, where the size of a graph is the number of edges in a graph. Notice that these Turán graphs have rigid structures, in particular, there are independent sets of size linear in n. It is then natural to ask for the size of an n-vertex K_{s+1}-free graph without these rigid structures, i.e. graphs with additional contstraints on their independence number. Such problems, first introduced by Sós [11] in 1969, are the substance of the Ramsey-Turán theory. Formally, given a graph H and natural numbers $m, n \in \mathbb{N}$, the *Ramsey-Turán number*, denoted by $\mathrm{RT}(n, H, m)$, is the maximum number of edges in an n-vertex H-free graph G with $\alpha(G) \le m$. Motivated by above reasons, the most classical case is when m is sublinear in n, i.e. $m = o(n)$.

Definition. *Given a graph H and $\delta \in (0, 1)$, let*

$$\varrho(H, \delta) := \lim_{n \to \infty} \frac{\mathrm{RT}(n, H, \delta n)}{n^2} \quad \text{and} \quad \varrho(H) := \lim_{\delta \to 0} \varrho(H, \delta). \tag{1.1}$$

We write

$$\mathrm{RT}(n, H, o(n)) = \varrho(H) \cdot n^2 + o(n^2).$$

We call $\varrho(H)$ the *Ramsey-Turán density* of H. The Ramsey-Turán density of cliques are well-understood. For odd cliques, Erdős and Sós [11] proved that $\varrho(K_{2s+1}) = \frac{1}{2}(\frac{s-1}{s})$ for all $s \ge 1$. The problem for even cliques is much harder. Szemerédi [25] first showed that $\varrho(K_4) \le \frac{1}{8}$. However no lower bound on $\varrho(K_4)$ was known until Bollobás and Erdős [6] provided a matching lower bound using an ingenious geometric construction, showing that $\varrho(K_4) = \frac{1}{8}$. Finally, Erdős, Hajnal, Sós and Szemerédi [9] determined the Ramsey-Turán density for all even cliques, proving that $\varrho(K_{2s}) = \frac{1}{2}(\frac{3s-5}{3s-2})$ for all $s \ge 2$.

While $\varrho(H)$ shows only the limit value, $\varrho(H, \delta)$ captures the transition behaviours of Ramsey-Turán number more accurately when independence number drops to $o(n)$. Capturing this more subtle behaviour, Fox, Loh and Zhao [13] proved that $\varrho(K_4, \delta) = \frac{1}{8} + \Theta(\delta)$. Building on Fox-Loh-Zhao's work, Lüders and Reiher [20] have very recently showed that, surprisingly, there is a precise

Date: October 20, 2018.

J.K. was supported by ERC grant 306349; Y.K. was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2017R1A6A3A04005963); and H.L. was supported by the Leverhulme Trust Early Career Fellowship ECF-2016-523.

formula for $\varrho(H,\delta)$ for all cliques and sufficiently small δ: for all $s \geq 2$, $\varrho(K_{2s-1},\delta) = \frac{1}{2}(\frac{s-2}{s-1}+\delta)$, while $\varrho(K_{2s},\delta) = \frac{1}{2}(\frac{3s-5}{3s-2}+\delta-\delta^2)$. Inspired by Lüders and Reiher's work, one of our results concerns the multicolour extension of this result. For more literature on Ramsey-Turán theory, we refer the readers to a survey of Simonovits and Sós [23]. See also [2, 3, 4] for more recent results on variants of Ramsey-Turán problem.

1.1. **Multicolour Ramsey-Turán problem.** Given graphs $H_1,\dots,H_k$, we say that a graph G is $(H_1,\dots,H_k)$*-free* if there exists an edge colouring $\phi : E(G) \to [k]$ such that for each $i \in [k]$, the spanning subgraph with all edges of colour i is H_i-free. Let $\mathrm{RT}(n, H_1,\dots,H_k, m)$ be the maximum size of an n-vertex $(H_1,\dots,H_k)$-free graph with independence number at most m, and define $\varrho(H_1,\dots,H_k,\delta)$ and $\varrho(H_1,\dots,H_k)$ analogous to (1.1). Erdős, Hajnal, Simonovits, Sós and Szemerédi [10] proved that the multicolour Ramsey-Turán density for cliques is determined by certain weighted Ramsey numbers (see Definition 5 and Theorem 2 in [10] for more details). Determining the actual values of $\varrho(K_{s_1},\dots,K_{s_k})$ turns out to be very difficult. Only sporadic cases are known [10]: $\varrho(K_3,K_3) = \frac{1}{4}$, $\varrho(K_3,K_4) = \frac{1}{3}$, $\varrho(K_3,K_5) = \frac{2}{5}$ and $\varrho(K_4,K_4) = \frac{11}{28}$. Even 2-coloured triangle versus a clique case, i.e. determining $\varrho(K_3,K_s)$, remains open. Recall that the Ramsey number $R(s,t)$ is the minimum integer N such that every blue/red colouring of K_N contains either a blue K_s or a red K_t. Erdős, Hajnal, Simonovits, Sós and Szemerédi [10] conjectured for all $s \geq 2$, $\varrho(K_3,K_{2s-1}) = \frac{1}{2}\left(1-\frac{1}{R(3,s)-1}\right)$.

Capturing more subtle behaviours of multicolour Ramsey-Turán number, Erdős and Sós [12] proved in 1979 that $\varrho(K_3,K_3,\delta) = \frac{1}{4}+\Theta(\delta)$ and conjectured that for sufficiently small δ, there exists $c > 0$ such that $\varrho(K_3,K_3,\delta) = \frac{1}{4}+c\delta$. In the following theorem, we determine the exact value of $\varrho(K_3,K_3,\delta)$ for all small $\delta > 0$, thus confirming the conjecture of Erdős and Sós. Furthermore, we also determine the exact values of $\varrho(K_3,K_4,\delta)$ and $\varrho(K_3,K_5,\delta)$. We remark that $\varrho(K_3,K_4,\delta)$ behaves quite differently from $\varrho(K_3,K_s,\delta)$ with $s \in \{3,5\}$. The extremal graph achieving the value of $\varrho(K_3,K_3,\delta)$ (resp. $\varrho(K_3,K_5,\delta)$) comes from taking the union of $T_2(n)$ (resp. $T_5(n)$) and F^*, certain almost δn-regular K_3-free graph with independence number at most δn. It turns out that the natural lower bound from the union of $T_3(n)$ and F^* is not optimal for $\varrho(K_3,K_4,\delta)$.

Theorem 1.1. *For sufficiently small $\delta > 0$, we have*
- $\varrho(K_3,K_3,\delta) = \frac{1}{4}+\frac{\delta}{2}$;
- $\varrho(K_3,K_4,\delta) = \frac{1}{3}+\frac{\delta}{2}+\frac{3\delta^2}{2}$;
- $\varrho(K_3,K_5,\delta) = \frac{2}{5}+\frac{\delta}{2}$;

We can see that the 2-colour Ramsey-Turán number $\varrho(K_3,K_s,\delta)$ shares some similarity with the single-colour problem $\varrho(K_s,\delta)$ as they both have an extra quadratic term when s is even. However, the single-colour Ramsey-Turán number has the same quadratic term for all even s. This is not the case for the 2-colour Ramsey-Turán number due to its relation to Ramsey number $R(3,\lceil s/2\rceil)$. Indeed, we give a construction showing that

$$\varrho(K_3,K_6,\delta) \geq \frac{5}{12}+\frac{\delta}{2}+2\delta^2.$$

We conjecture that the equality above holds (see concluding remark for more details).

In the following theorem, we determine Ramsey-Turán numbers for (K_3,K_s) for all $s \geq 3$ when the independence number condition is slightly more strict than sublinear, providing evidence towards the Erdős-Hajnal-Simonovits-Sós-Szemerédi conjecture. Let $\omega(n)$ be a function growing to infinity arbitrarily slowly as $n \to \infty$. For each integer $s \geq 2$, define

$$g_s^{\omega}(n) := \frac{n}{e^{\omega(n)\cdot(\log n)^{1-1/s}}}. \tag{1.2}$$

We omit ω and write $g_s(n)$ whenever the result holds for any function $\omega(n)$ growing to infinity. Note that $n \gg g_s(n) \gg n^{1-\varepsilon}$, for any $\varepsilon > 0$.

Theorem 1.2. *For all $s \geq 2$, we have*

- $\mathrm{RT}(n, K_3, K_{2s-1}, g_s(n)) = \frac{1}{2}\left(1 - \frac{1}{R(3,s)-1}\right) n^2 + o(n^2)$*; and*
- $\mathrm{RT}(n, K_3, K_{2s}, g_s(n)) = \frac{1}{2}\left(1 - \frac{1}{R(3,s)}\right) n^2 + o(n^2)$.

1.2. **Phase transition.** Our next result concerns phase transitions of the single-coloured Ramsey-Turán number. A graph H has *Ramsey-Turán phase transition at f* if

$$\lim_{n\to\infty} \frac{\mathrm{RT}(n,H,f(n)) - \mathrm{RT}(n,H,o(f(n)))}{n^2} > 0,$$

where $\mathrm{RT}(n,H,o(f(n)) = \lim_{\delta\to 0}\mathrm{RT}(n,H,\delta\cdot f(n))$. In other words, a slightly stronger upper bound on the independence number, $o(f(n))$ instead of $f(n)$, would result in a drop at the maximum possible edge-density of an H-free graph (see [1] for more details).

From odd cliques, the result of Erdős-Sós [11] shows that K_{2s+1}, with $s \geq 1$, has its first phase transition at $f(n) = n$, where the density drops from $\frac{1}{2}(\frac{2s-1}{2s})$ to $\frac{1}{2}(\frac{s-1}{s})$. In fact,

$$\mathrm{RT}(n, K_{2s+1}, c\sqrt{n\log n}) = \frac{1}{2}\left(\frac{s-1}{s} + o(1)\right) n^2 \quad \text{for} \quad c > \frac{2}{\sqrt{s}}.$$

Balogh, Hu and Simonovits ([1], Theorem 2.7) proved that

$$\mathrm{RT}(n, K_{2s+1}, o(\sqrt{n\log n})) \leq \frac{1}{2}\left(\frac{2s-3}{2s} + o(1)\right) n^2,$$

showing that the second phase transition happens at $f(n) = \sqrt{n\log n}$ (around the inverse function of $R(3,n)$). Erdős and Sós [11] asked whether $\mathrm{RT}(n,K_5,o(\sqrt{n})) = o(n^2)$. Sudakov [24] showed that it is true if a slightly stronger bound is imposed on the independence number: $\mathrm{RT}(n,K_5,g_2(\sqrt{n})) = o(n^2)$. Later, Balogh, Hu, Simonovits [1] answered Erdős and Sós's question in a stronger form, showing that: $\mathrm{RT}(n,K_5,o(\sqrt{n\log n})) = o(n^2)$.

The situation for even cliques, K_{2s} with $s \geq 2$, is again less clear apart from the first phase transition at $f(n) = n$ as shown by Erdős-Hajnal-Simonovits-Sós-Szemerédi [10], where the density decreases from $\frac{1}{2}(\frac{2s-2}{2s-1})$ to $\frac{1}{2}(\frac{3s-5}{3s-2})$. Extending a result of Sudakov [24], Balogh, Hu, Simonovits ([1], Theorem 3.4) showed that:

$$\mathrm{RT}(n, K_{2s}, f(n)) = \frac{1}{2}\left(\frac{s-2}{s-1} + o(1)\right) n^2$$

for any $c\sqrt{n\log n} < f(n) \leq g_s(n)$ where $c > 2/\sqrt{s-1}$; while Fox, Loh and Zhao [13] showed that

$$\mathrm{RT}(n, K_{2s}, g^*(n)) = \frac{1}{2}\left(\frac{3s-5}{3s-2} + o(1)\right) n^2,$$

where $g^*(n) := ne^{-o\left(\sqrt{\frac{\log n}{\log\log n}}\right)}$. Thus, the second phase transition for K_{2s} happens somewhere in the small window between $g^*(n)$ and $g_s(n)$. The third phase transition for even cliques occurs at $f(n) = \sqrt{n\log n}$, but not a single extremal density is known except the trivial case of K_4. For example, $\mathrm{RT}(n,K_6,o(\sqrt{n\log n})) \leq \frac{n^2}{6} + o(n^2)$ and we do not know whether it is $o(n^2)$. For K_8, Balogh, Hu and Simonovits [1] showed that

- $\mathrm{RT}(n,K_8,c\sqrt{n\log n}) = \frac{n^2}{3} + o(n^2)$ for $c > 2/\sqrt{3}$;
- $\frac{2n^2}{7} + o(n^2) \geq \mathrm{RT}(n,K_8,o(\sqrt{n\log n})) \geq \mathrm{RT}(n,K_7,o(\sqrt{n\log n})) = \frac{n^2}{4} + o(n^2)$;
- $\mathrm{RT}(n,K_8,g_2(\sqrt{n})) = \frac{n^2}{4} + o(n^2)$,

and raised the question of whether

$$\mathrm{RT}(n,K_8,o(\sqrt{n\log n})) = \mathrm{RT}(n,K_7,o(\sqrt{n\log n})).$$

So the Ramsey-Turán density for K_8 drops from $1/3$ to at most $2/7$ around $\sqrt{n\log n}$. It is not clear when in between $o(\sqrt{n\log n})$ and $g_2(\sqrt{n})$, it drops to $1/4$. In the following theorem, we close this gap, proving that

$$\mathrm{RT}(n, K_8, o(\sqrt{n\log n})) = \frac{n^2}{4} + o(n^2).$$

This answers Balogh-Hu-Simonovits's question positively and provides the first exact value of non-trivial extremal density for the third phase transition of an even clique.

Theorem 1.3. *For any $\gamma > 0$, there exists $\delta > 0$ such that the following holds. Let G be an n-vertex K_8-free graph with $\alpha(G) \le \delta\sqrt{n\log n}$. Then $e(G) \le \frac{n^2}{4} + \gamma n^2$.*

Organisation of the paper. In Section 2, we give preliminaries necessary for the proofs. Then we present the proofs of Theorem 1.3 in Section 3, and the lower bounds in Theorem 1.1 in Section 4. We then prove the upper bounds in Theorem 1.1 in Sections 5 and 6. The proof of Theorem 1.2 will be given in Section 7. Finally in Section 8, we make some concluding remarks.

2. Preliminaries

In this section, we introduce some notation, tools and lemmas. Denote $[q] := \{1, 2, \dots, q\}$, $[p, q] := \{p, p+1, \dots, q\}$, and $\binom{X}{i}$ (resp. $\binom{X}{\le i}$) denotes the set of all subsets of a set X of size i (resp. at most i). We may abbreviate a singleton $\{x\}$ (resp. a pair $\{x, y\}$) as x (resp. xy). If we claim that a result holds whenever $0 < b \ll a \ll 1$, this means that there are a constant $a_0 \in (0, 1)$ and a non-decreasing function $f : (0, 1) \to (0, 1)$ (that may depend on any previously defined constants or functions) such that the result holds for all $a, b \in (0, 1)$ with $a \le a_0$ and $b \le f(a)$. We write $a = b \pm c$ if $b - c \le a \le b + c$. We may omit floors and ceilings when they are not essential.

Let $G = (V, E)$ be a graph and $A, B, V_1, \dots, V_p \subseteq V$. Denote by $\overline{A} := V \setminus A$ the complement of A. Let $G[A] := (A, \{xy \in E : x, y \in A\})$ denote the induced subgraph of G on A, and denote by $N(A, B)$ the common neighbourhood of A in B. Write $N(v, B)$ instead of $N(\{v\}, B)$, and $d(v, B) = |N(v, B)|$. Denote by $G[V_1, \dots, V_p]$ the p-partite subgraph of G induced by p-partition $V_1 \cup \dots \cup V_p$. We say that a partition $U_1 \cup \ldots \cup U_p$ of V is a *max-cut p-partition* of G if $e(G[U_1, \dots, U_p])$ is maximised among all p-partition of V. Denote by $\delta^{\mathrm{cr}}(G[V_1, \dots, V_p]) := \min_{ij \in \binom{[p]}{2},\ v \in V_i} d(v, V_j)$ the *minimum crossing degree* of G with respect to the partition $V_1 \cup \ldots \cup V_p$. For each $n, p \in [n]$, $T_p(n)$ denotes the n-vertex Turán graph, which is the n-vertex complete p-partite graph such that each partite set has size either $\lfloor n/p \rfloor$ or $\lceil n/p \rceil$. For two n-vertex graphs G and H, we define $|G \triangle H|$ be the minimum number $N = N_1 + N_2$ such that we can obtain a graph isomorphic to H after deleting N_1 edges from G and adding N_2 edges to G.

Given $\phi : E(G) \to [k]$, throughout the paper, for each $i \in [k]$, we will always denote G_i the spanning subgraph of G induced by all edges of colour i. We say that ϕ, and also G, is $(K_{s_1}, \dots, K_{s_k})$-*free* if G_i is K_{s_i}-free for each $i \in [k]$. We will write $\phi(A, B) = i$ if $\phi(e) = i$ for all $e \in E(G[A, B])$, and write $\phi(v, B)$ instead of $\phi(\{v\}, B)$. If ϕ is also defined on $V(G)$, we write $\phi(A) = i$ if $\phi(v) = i$ for all $v \in A$. The following result will be useful.

Theorem 2.1 ([16])**.** *Let G be an n-vertex K_4-free graph with $e(G) \ge n^2/4 + t$. Then G contains at least t edge-disjoint triangles.*

Given $d, n \in \mathbb{N}$, denote by $F(n, d)$ an n-vertex d-regular triangle-free graph with $\alpha(G) = d$. Let $\mathcal{B} \subseteq (0, 1)$ consists of all the rationals δ for which there exists some $F(n, d)$ with $d/n = \delta$. We will use a result of Brandt [7], which states that $\mathcal{B}$ is dense in $(0, 1/3)$, in the following form.

Theorem 2.2 ([7])**.** *For any $0 < \eta, \delta < 1/3$, there exists $n_0 > 0$ such that the following holds for all $n \ge n_0$. For some $d \in [(\delta - \eta)n, \delta n]$, there exists a graph $F(n, d)$.*

The following is a result of Füredi proving stability of K_{p+1}-free graphs.

Theorem 2.3 ([15]). *Suppose that $t \in \mathbb{N}$ and G is an n-vertex K_{p+1}-free with $e(G) \ge e(T_{n,p}) - t$. Then there exists $v_1, \dots, v_p$ such that $e(K_{v_1,\dots,v_p}) \ge e(T_{n,p}) - 2t$ and $|G \triangle K_{v_1,\dots,v_p}| \le 3t$. Consequently, $v_i = n/p \pm 2\sqrt{t}$ for all $i \in [p]$ and $|G \triangle T_p(n)| = O(\sqrt{t}n)$.*

The following theorem follows from Shearer's bound on Ramsey number $R(3,k) \le (1+o(1))k^2/\log k$ (see also [5, 8, 21] for more recent development on $R(3,k)$).

Theorem 2.4. [22] *There exists $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$, any graph on at least $2k^2/\log k$ vertices contains either a triangle or an independent set of size k.*

We will make use of the multicolour version of the Szemerédi Regularity Lemma (see, for example, [18, Theorem 1.18]). We introduce the relevant definitions. Let $X, Y \subseteq V(G)$ be disjoint non-empty sets of vertices in a graph G. The *density* of (X,Y) is $d_G(X,Y) := \frac{e(G[X,Y])}{|X||Y|}$. For $\varepsilon > 0$, the pair (X,Y) is *ε-regular* in G if for every pair of subsets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \ge \varepsilon|X|$ and $|Y'| \ge \varepsilon|Y|$, we have $|d_G(X,Y) - d_G(X',Y')| \le \varepsilon$. Additionally, if $d_G(X,Y) \ge \gamma$, for some $\gamma > 0$, we say that (X,Y) is *(ε,γ)-regular*. A partition $V(G) = V_0 \cup V_1 \cup \ldots \cup V_m$ is an *ε-regular partition* of a k-edge-coloured graph G if

(1) $|V_0| \le \varepsilon|V(G)|$ and for all $ij \in \binom{[m]}{2}$, $|V_i| = |V_j|$;
(2) for each $i \in [m]$, all but at most εm choices of $j \in [m]$ satisfies that the pair (V_i, V_j) is ε-regular in G_ℓ for each colour $\ell \in [k]$.

Lemma 2.5 (Multicolour Regularity Lemma [18]). *Suppose $0 < 1/M' \ll \varepsilon, 1/M \ll 1/k \le 1$ and $n \ge M$. Suppose G is an n-vertex k-edge-coloured graph and $U_1 \cup U_2$ is a partition of $V(G)$. Then there exists an ε-regular partition $V(G) = V_0 \cup V_1 \cup \ldots \cup V_m$ with $M \le m \le M'$ such that for each $i \in [m]$ we have either $V_i \subseteq U_1$ or $V_i \subseteq U_2$.*

Given $\varepsilon, \gamma > 0$, a graph G, a colouring $\phi : E(G) \to [k]$ and a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_m$, define the *reduced graph* $R := R(\varepsilon, \gamma, \phi, (V_i)_{i=1}^m)$ of order m as follows: $V(R) = [m]$ and $ij \in E(R)$ if (V_i, V_j) is (i) ε-regular with respect to G_p for every $p \in [k]$ and (ii) $d(G_q[V_i, V_j]) \ge \gamma$ for some $q \in [k]$. For brevity, we may omit ϕ or $(V_i)_{i=1}^r$ in the notation when these are clear. It is easy to see that we have

$$e(G) \le e(R) \cdot \left(\frac{n}{m}\right)^2 + \frac{n^2}{m} + k\gamma n^2. \tag{2.1}$$

Given a graph R and $s \in \mathbb{N}$, let $R(s)$ be the graph obtained by replacing every vertex of R with an independent set of size s and replacing every edge of R with $K_{s,s}$. The following lemmas provide some useful properties related to regular partitions.

Lemma 2.6 (Counting Lemma, Theorem 2.1 in [18]). *Suppose $0 < 1/n \ll \varepsilon \ll \gamma, 1/h \le 1$. Suppose that H is an h-vertex graph and R is a graph such that $H \subseteq R(h)$. If G is a graph obtained by replacing every vertex of R with an independent set of size n and replacing every edge of R with an (ε,γ)-regular pair, then G contains at least $(\gamma/2)^{e(H)} N^{|V(H)|}$ copies of H.*

Lemma 2.7 (Slicing Lemma, Fact 1.5 in [18]). *Let $\varepsilon < \alpha, \gamma, 1/2$. Suppose that (A,B) is an (ε,γ)-regular pair in a graph G. If $A' \subseteq A$ and $B' \subseteq B$ satisfy $|A'| \ge \alpha|A|$ and $|B'| \ge \alpha|B|$, then (A', B') is an $(\varepsilon', \gamma - \varepsilon)$-regular pair in G, where $\varepsilon' := \max\{\varepsilon/\alpha, 2\varepsilon\}$.*

Lemma 2.8 (Claim 7.1 in [1] with $p = 2$). *For a given function g_s as in* (1.2)*, suppose $0 < 1/n \ll 1/m, \varepsilon \ll \gamma < 1$. Suppose that G is an n-vertex graph with $\alpha(G) \le g_s(n)$ and $V_0 \cup V_1 \cup \cdots \cup V_m$ is an ε-regular partition and $R = R(\varepsilon,\gamma)$ is a corresponding reduced graph. If $K_s \subseteq R$, then we have $K_{2s} \subseteq G$.*

The following lemma will be useful to guarantee a certain minimum degree condition in a dense graph.

Lemma 2.9. *Suppose $0 < 1/n \ll \varepsilon \ll d \leq 1$. Suppose that G is an n-vertex graph with $e(G) \geq (d+\varepsilon)n^2/2$. Then G contains an n'-vertex subgraph G' with $n' \geq \varepsilon^{1/2}n/2$ such that $e(G') \geq (dn'^2 + \varepsilon n^2 - d(n-n'))/2$ and $\delta(G') \geq dn'$.*

Proof. Obtain a sequence of graphs $G_n := G, G_{n-1}, \dots$ as follows. For each $i \leq n$, if there exists a vertex $v_i \in V(G_i)$ with $d_{G_i}(v_i) \leq di$, then set $G_{i-1} := G_i \setminus \{v_i\}$. Let $G_{n'}$ be the final graph. Then we have

$$n'^2 \geq 2e(G_{n'}) > (d+\varepsilon)n^2 - \sum_{i=n'+1}^{n} 2di = \varepsilon n^2 + dn'^2 - dn + dn'.$$

This implies $n' \geq \varepsilon^{1/2}n/2$, thus proving the lemma. □

Note that, for an n-vertex 2-edge-coloured graph G with $\alpha(G) = o(n)$, both G_1, G_2 can have $\Omega(n)$ independence number. We will use the following lemma combined with regularity lemma to obtain a regular partition such that each part of the partition induces a graph with small independence number in one of the two colours.

Lemma 2.10 (Lemma 3.1 in [2] with $r=2$)**.** *Let $c > 0$, G be an n-vertex graph with $\alpha(G) \leq c^2 n$ and $\phi : E(G) \to [2]$. Then there exists a partition $V(G) = V_1^* \cup V_2^*$*[1] *such that for every $i \in [2]$, $\alpha(G_i[V_i^*]) \leq cn$.*

3. Proof of Theorem 1.3

We need first the following variation of dependent random choice lemma. For more on the dependent random choice method, we refer the readers to a survey of Fox and Sudakov [14].

Lemma 3.1. *Let $0 < 1/n \ll \gamma < 1$ and G be a 3-partite graph with vertex partition $Z_1 \cup Z_2 \cup Z_3$ such that $|Z_i| = n$ for each $i \in [3]$. If $d(v, Z_i) \geq \gamma n$ holds for all $v \in Z_1$ and $i \in \{2,3\}$, then there exists a set $S \subseteq Z_1$ of size $\frac{1}{2}n^{2/3}$ such that every pair of vertices $P \in \binom{S}{2}$ satisfies $|N(P, Z_i)| \geq \gamma^9 n$ for each $i \in \{2,3\}$.*

Proof. Set $q := \frac{\log n}{6\log(1/\gamma)}$. For each $i \in \{2,3\}$, pick q vertices in Z_i uniformly at random with repetition and denote by Q_i the set of chosen vertices. We call a pair $P \in \binom{Z_1}{2}$ *bad* if there exists $i \in \{2,3\}$ such that $|N(P,Z_i)| < \gamma^9 n$. Define $S' := N(Q_2 \cup Q_3, Z_1)$, and define a random variable X as the number of bad pairs in S'. Note that for each bad pair $P \in \binom{Z_1}{2}$, we have

$$\mathbb{P}[P \subseteq S'] = \mathbb{P}[Q_2 \cup Q_3 \subseteq N(P)] = \prod_{i=2}^{3} \left(\frac{|N(P,Z_i)|}{|Z_i|} \right)^q \leq \gamma^{18q}.$$

Thus, by the linearity of expectation, we have that $\mathbb{E}[X] \leq \binom{|Z_1|}{2} \cdot \gamma^{18q} \leq n^2 \gamma^{18q}$. On the other hand,

$$\mathbb{E}|S'| = \sum_{v \in Z_1} \mathbb{P}[v \in S'] = \sum_{v \in Z_1} \mathbb{P}[Q_2 \cup Q_3 \subseteq N(v)] = \sum_{v \in Z_1} \prod_{i=2}^{3} \left(\frac{d(v,Z_i)}{|Z_i|} \right)^q \geq n\gamma^{2q}.$$

So, there exists choices of Q_2 and Q_3 such that $\mathbb{E}[|S'| - X] \geq n\gamma^{2q} - n^2\gamma^{18q} \geq \frac{1}{2}n^{2/3}$. Then the set S obtained from deleting one vertex from every bad pair in S' has the desired properties. □

Proof of Theorem 1.3. Fix constants δ, M', ε as follows:

$$0 < 1/n_0 \ll \delta < 1/M' \ll \varepsilon \ll \gamma \ll 1.$$

Assume that G is an n-vertex graph with $n \geq n_0$ and $\alpha(G) \leq \delta\sqrt{n \log n}$. Apply the regularity lemma (Lemma 2.5 with $k=1$) with $G, V(G), \emptyset, \varepsilon, \varepsilon^{-1}, 1$ and M' playing the roles of $G, U_1, U_2, \varepsilon, M, k$

[1] It could be that some V_i^* is empty.

and M', respectively to obtain a regularity partition $V_0 \cup V_1 \cup \cdots \cup V_m$ and the reduced graph $R = R(\varepsilon, \gamma/2)$ of order m with $\varepsilon^{-1} \leq m \leq M'$. Note that R is K_4-free by Lemma 2.8. We say that a triangle ijk in R is *chubby* if $d_G(V_{i'}, V_{j'}) \geq 2/3 + \gamma$ for some $i'j' \in \binom{\{i,j,k\}}{2}$. We first show that there is no chubby triangle in R.

Claim 3.2. *No triangle in R is chubby.*

Proof. Suppose without loss of generality that $\{1, 2, 3\}$ induces a triangle in R with $d(V_2, V_3) \geq 2/3 + \gamma$. By the definition of regular pair, it is well-known that for each $i \in [3]$, there exists a subset $V_i^* \subseteq V_i$ such that $|V_i^*| = (1 - 2\varepsilon)|V_i|$ and $\delta^{\mathrm{cr}}(G[V_1^*, V_2^*, V_3^*]) \geq \gamma|V_i^*|/3$. Applying Lemma 3.1 to $G[V_1^*, V_2^*, V_3^*]$ with V_i^*s playing the roles of Z_is, we obtain a set $S \subseteq V_1^*$ of size at least $\frac{1}{3}\left(\frac{n}{m}\right)^{2/3}$ such that every pair $P \in \binom{S}{2}$ satisfies $|N(P, V_i^*)| \geq (\frac{\gamma}{3})^9 n/m$ for each $i \in \{2, 3\}$. As $\frac{1}{3}\left(\frac{n}{m}\right)^{2/3} \geq \alpha(G)$, the set S contains an edge $uv \in E(G)$ with $|N(uv, V_i^*)| \geq (\frac{\gamma}{3})^9 n/m$ for each $i \in \{2, 3\}$. Using Lemma 2.7 and again deleting low degree vertices, we get $V_i' \subseteq N(uv, V_i^*)$ for $i \in \{2, 3\}$ such that $|V_2'| = |V_3'| \geq \gamma^{10} n/m$; (V_2', V_3') is $(\sqrt{\varepsilon}, \frac{2}{3} + \frac{\gamma}{4})$-regular with $\delta(G[V_2', V_3']) \geq (\frac{2}{3} + \frac{\gamma}{5})|V_3'|$. Observe that V_2' must contain a triangle, as otherwise Theorem 2.4 implies that there exists an independent set of size at least

$$\frac{1}{2}\sqrt{|V_2'| \log |V_2'|} \geq \frac{1}{2}\sqrt{\gamma^{10}\frac{n}{m} \log\left(\gamma^{10}\frac{n}{m}\right)} > \frac{1}{m}\sqrt{n \log n} > \delta\sqrt{n \log n} \geq \alpha(G), \tag{3.1}$$

a contradiction. Let T be a triangle in V_2'. Then we have that

$$|N(T, V_3')| \geq 3\delta(G[V_2', V_3']) - 2|V_3'| \geq \frac{\gamma}{2}|V_3'| \geq \gamma^{12}\frac{n}{m}.$$

Almost identical calculation as (3.1) shows that $N(T, V_3')$ contains a triangle, which together with u, v and T forms a copy of K_8, a contradiction. □

We are now ready to prove Theorem 1.3. Let $t \in \mathbb{R}$ be such that $e(R) = m^2/4 + t$. If $t < 0$, then (2.1) with the definition of $R = R(\varepsilon, \gamma/2)$ implies that

$$e(G) \leq (m^2/4)(n/m)^2 + n^2/m + \gamma n^2/2 \leq n^2/4 + \gamma n^2$$

as $1/m \ll \gamma$. We may thus assume $t \geq 0$.

Recall that R is K_4-free, Turán's theorem implies that $e(R) \leq m^2/3$ and $t/m^2 \leq 1/12$; and by Lemma 2.1, $E(R)$ can be decomposed into $m^2/4 - 2t$ edges and t edge-disjoint triangles. Each triangle in R, by Claim 3.2, corresponds to at most

$$3 \cdot (2/3 + \gamma) \cdot (n/m)^2 = (2 + 3\gamma)(n/m)^2$$

edges in G. Hence

$$e(G) \leq \left(\frac{m^2}{4} - 2t + (2 + 3\gamma)t\right)\frac{n^2}{m^2} + \frac{n^2}{m} + \frac{\gamma}{2}n^2 \leq \frac{n^2}{4} + \gamma n^2,$$

as desired. □

4. Lower bound constructions for $\varrho(K_3, K_s, \delta)$

For each $s \in \{3, 4, 5\}$ and small $\delta > 0$, we will construct an n-vertex (K_3, K_s)-free graph G with $\alpha(G) \leq \delta n$ and the desired edge-density. This provides a lower bound on $\varrho(K_3, K_s, \delta)$. Throughout this section, we use $X_1, \ldots, X_k$ for the partite sets of $T_k(n)$.

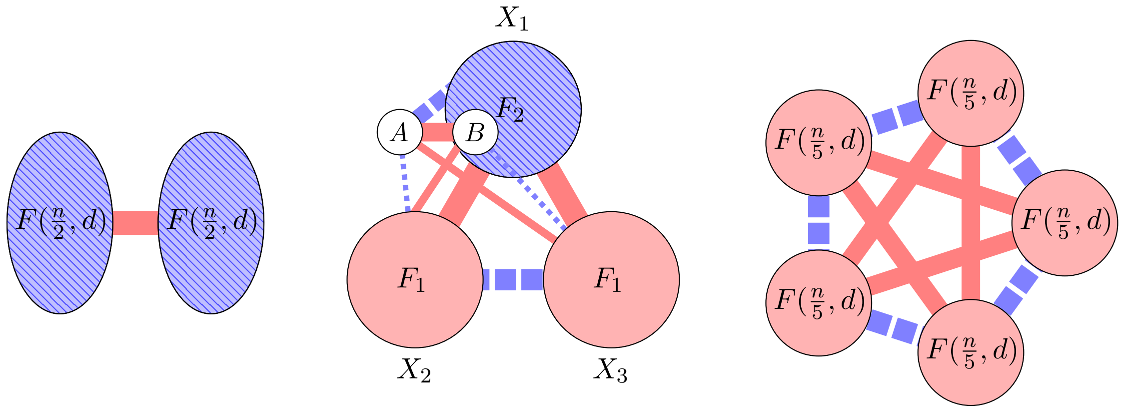

Figure 1: Constructions for $s = 3, 4, 5$ in order. Colour 1: dotted blue. Colour 2: red.

4.1. **Lower bound for $\varrho(K_3, K_s, \delta)$ when $s \in \{3, 5\}$.** If $s = 3$, let G be a graph obtained from putting a copy of $F(\frac{n}{2}, d)$, for some $d \in [\delta n - o(n), \delta n]$, in both partite sets of $T_2(n)$. It is easy to see that $\alpha(G) \leq \delta n$ and $e(G) = \frac{n^2}{4} + \frac{\delta n^2}{2} + o(n^2)$. It is also easy to check that the following edge-colouring ϕ is a (K_3, K_3)-free colouring: $\phi(e) = 1$ for all $e \in \cup_{i\in[2]} G[X_i]$; and $\phi(X_1, X_2) = 2$, see Figure 1.

If $s = 5$, let G be a graph obtained from putting a copy of $F(\frac{n}{5}, d)$, for some $d \in [\delta n - o(n), \delta n]$, in each partite set of $T_5(n)$. It is easy to see that $\alpha(G) \leq \delta n$ and $e(G) = \frac{2n^2}{5} + \frac{\delta n^2}{2} + o(n^2)$. It is also easy to check that the following edge-colouring ϕ is a (K_3, K_5)-free colouring: $\phi(e) = 2$ for all $e \in \cup_{i\in[5]} G[X_i]$; $\phi(X_i, X_{i+1}) = 2$ for all $i \in [5]$ (addition modulo 5); and all other edges are of colour 1, see Figure 1.

4.2. **Lower bound for $\varrho(K_3, K_4, \delta)$.** We construct an n-vertex (K_3, K_4)-free graph G with $\alpha(G) \leq \delta n$ and $(\frac{1}{3} + \frac{\delta}{2} + \frac{3\delta^2}{2} - o(1))n^2$ edges as follows.

- By Theorem 2.2, there exist $F_1 := F(\frac{n}{3}, d_1)$ and $F_2 := F(\frac{n}{3} - \delta n, d_2)$ where $d_i \in [\delta n - o(n), \delta n]$ for each $i \in [2]$. So $e(F_1) = \frac{\delta n^2}{6} + o(n^2)$ and $e(F_2) = \frac{\delta n^2}{6} - \frac{\delta^2 n^2}{2} + o(n^2)$.
- Let $B = \{b_1, b_2, \dots, b_{d_2}\}$ be an independent set of size d_2 in F_2. Let F be an $n/3$-vertex graph obtained from F_2 by
  - first adding a clone set of B, i.e. a set of d_2 new vertices $A = \{a_1, a_2, \dots, a_{d_2}\}$ with $N_F(a_i) := N_{F_2}(b_i)$ for each $i \in [d_2]$;
  - adding all $[A, B]$-edges; and
  - adding an additional set of $\delta n - d_2$ isolated vertices.

  Note that F is *not* triangle-free, and
$$e(F) = e(F_2) + d_2^2 + d_2^2 = \frac{\delta n^2}{6} + \frac{3\delta^2 n^2}{2} + o(n^2).$$
- Finally, let G be the graph obtained from $T_3(n)$ on partite sets X_i, $i \in [3]$, by putting a copy of F in X_1 and a copy of F_1 in X_2 and X_3.

It is clear that G has the desired size and easy to check that the following 2-edge-colouring ϕ of G is (K_3, K_4)-free, see Figure 1:

- let $\phi(A, X_2) = \phi(B, X_3) = \phi(X_2, X_3) = 1$;
- let $\phi(e) = 1$, for all $e \in E(G[X_1] \setminus [A, B])$;
- all other edges are of colour 2.

5. Upper bound for $\varrho(K_3, K_4, \delta)$

For the convenience of the reader, we rephrase the upper bound as follows.

Lemma 5.1. *Suppose* $0 < 1/n \ll \delta \ll 1$. *Let* G *be an* n*-vertex* (K_3, K_4)*-free graph with* $\alpha(G) \le \delta n$. *Then*

$$e(G) \le \frac{n^2}{3} + \frac{\delta n^2}{2} + \frac{3\delta^2 n^2}{2}.$$

We use the stability approach. A weak stability was proven in [10][2], stating that an n-vertex (K_3, K_4)-free graph G with $\alpha(G) \le \delta n$ is close to $T_3(n)$. For the exact result for $\varrho(K_3, K_4, \delta)$, we need a coloured stability, which roughly says that any (K_3, K_4)-free colouring of an almost extremal graph should look similar to the colouring given in the lower bound construction.

5.1. Coloured stability.

Lemma 5.2. *Suppose* $0 < 1/n \ll \delta \ll \gamma < 1$. *Let* G *be an* n*-vertex* (K_3, K_4)*-free graph with* $\alpha(G) \le \delta n$ *and* $\delta(G) \ge 2n/3$. *Then for any* (K_3, K_4)*-free* 2*-edge colouring* $\phi : E(G) \to [2]$, *there exists a partition* $X_1 \cup X_2 \cup X_3$ *of* $V(G)$ *such that the following holds.*

(A1) *For each* $i \in [3]$, *we have* $|X_i| = n/3 \pm 2\gamma^{1/7} n$.
(A2) $\alpha(G_1[X_1]) \le \gamma^{1/4} n$.
(A3) *For each* $i \in [3]$, *we have* $\Delta(G[X_i]) \le \gamma^{1/18} n$.
(A4) *For each* $v \in X_1$, *we have* $\min\{d_{G_1}(v, X_2),\ d_{G_1}(v, X_3)\} < \gamma^{1/9} n$.
(A5) $\delta^{\mathrm{cr}}(G[X_1, X_2, X_3]) \ge n/3 - \gamma^{1/19} n$.
(A6) *For all* $i \in \{2, 3\}$ *and* $v \in X_i$, *we have* $d_{G_2}(v, X_1) \ge |X_1| - \gamma^{1/20} n$.

Furthermore, one of the following occurs:

(P1) *For all* $i \in \{2, 3\}$ *and* $v \in X_i$, *we have* $\alpha(G_2[X_i]) \le \gamma^{1/4} n$ *and* $d_{G_1}(v, X_{5-i}) \ge |X_{5-i}| - \gamma^{1/20} n$.
(P2) *For all* $i \in \{2, 3\}$ *and* $v \in X_i$, *we have* $\alpha(G_1[X_i]) \le \gamma^{1/4} n$ *and* $d_{G_2}(v, X_{5-i}) \ge |X_{5-i}| - \gamma^{1/20} n$.

We need an additional definition for the proof of Lemma 5.2.

Definition 5.3. *Given a weighted graph* R *with weight* $d : E(R) \to (0, 1]$ *and* $Y \subseteq X \subseteq V(R)$, *a* γ-generalised clique *of order* $t := |X| + |Y|$, *denoted by* Z_t, *on* (X, Y) *is a clique on* X *with* $d(e) > 1/2 + \gamma$ *for every* $e \in E(R[Y])$.

For brevity, we will write $(a_1 a_2 \dots a_s, b_1 b_2 \dots b_{s'})$ for $(\{a_1, \dots, a_s\}, \{b_1, \dots, b_{s'}\})$.

Proof of Lemma 5.2. We choose the constants as follows:

$$0 < 1/n \ll \delta \ll \delta^* \ll 1/m \ll \varepsilon \ll \gamma \ll 1.$$

We apply Lemma 2.10 with $c = \delta^{1/2}$ to obtain a partition $V_1^* \cup V_2^*$ such that $\alpha(G_i[V_i^*]) \le \delta^{1/2} n$. Apply Theorem 2.5 with $G, V_1^*, V_2^*, \phi, \varepsilon, \varepsilon^{-1}$ and M' playing the roles of $G, U_1, U_2, \phi, \varepsilon, M$ and M' to obtain an ε-regular partition $V_0 \cup V_1 \cup \dots \cup V_m$ with $\varepsilon^{-1} \le m \le M'$ which refines the partition $V_1^* \cup V_2^*$. Let $R := R(\varepsilon, \gamma, \phi, (V_i)_{i \in [m]})$ be its reduced graph. It was shown in [10] (Theorem 3(b) and (e)) that $|G \triangle T_3(n)| \le \delta^* n^2$. As a consequence, the number of K_4 in G is at most $\delta^* n^4$. It is well-known that the reduced graph R essentially inherits the structure of G: $\delta(R) \ge (2/3 - 3\gamma) m$ and R is K_4-free. Indeed, if $K_4 \subseteq R$, then by Lemma 2.6, G contains at least $(\gamma/2)^6 (n/m)^4 / 2 > \delta n^4$ copies of K_4, a contradiction. Thus, by Theorem 2.3,

$$|R \triangle T_3(m)| \le \gamma^{1/3} m^2. \tag{5.1}$$

We define a colouring $\phi^{\mathrm{ind}} : V(R) \cup E(R) \to [2]$, induced by ϕ, as follows:

(i) for each $k \in [m]$, we have $\phi^{\mathrm{ind}}(k) = i$ if $V_k \subseteq V_i^*$; and
(ii) for each $pq \in E(R)$, we have $\phi^{\mathrm{ind}}(pq) = 1$ if $d_{G_1}(V_p, V_q) \ge \gamma$, and $\phi^{\mathrm{ind}}(pq) = 2$ if $d_{G_1}(V_p, V_q) < \gamma$ and $d_{G_2}(V_p, V_q) \ge \gamma$.

[2] Their proof missed a case, which can be easily fixed. We include it in the online arXiv version.

We remark that colour 1 has "higher priority" on $E(R)$ in ϕ^{ind}, i.e. if (V_i, V_j) is dense in both G_1 and G_2, then we have $\phi^{\text{ind}}(ij) = 1$. This asymmetry is needed for the embedding later. For each pq, we let $d(pq) := d_{G_{\phi^{\text{ind}}(pq)}}(V_p, V_q)$ be the weight on $E(R)$, and we consider R as a weighted graph. It is also well-known that for each $p \in V(R)$, we have

$$\sum_{q\in N_R(p)} d(pq) \geq \left(\frac{m}{n}\right)^2 \left(\delta(G)\frac{n}{m} - \frac{\varepsilon n^2}{m} - \frac{2\gamma n^2}{m} - \left(\frac{n}{m}\right)^2\right) \geq (2/3 - 3\gamma)m. \tag{5.2}$$

Let R' be the graph obtained from R by deleting all edges of weight at most $1/2+\gamma$. Then for each $p \in V(R)$, we have

$$(2/3 - 3\gamma)m \leq \sum_{q\in N_R(p)} d(pq) \leq d_{R'}(p) + (1/2+\gamma)(d_R(p) - d_{R'}(p)),$$

thus, as $\delta(R) \geq (2/3 - 3\gamma)m$, we have

$$d_{R'}(p) \geq 4m/3 - d_R(p) - 9\gamma m. \tag{5.3}$$

Moreover, by (5.1), we know $e(R) \leq \frac{m^2}{3} + \gamma^{1/3}m^2$. As $\delta(G) \geq 2n/3$, similar to (2.1), we have

$$n^2/3 \leq e(G) \leq (e(R) - e(R'))\frac{(1/2+\gamma)n^2}{m^2} + e(R')\frac{n^2}{m^2} + (2\gamma + \varepsilon + 1/m)n^2.$$

This implies

$$e(R) - e(R') \leq \gamma^{1/4}m^2. \tag{5.4}$$

We will omit γ in the term 'γ-generalised clique'. For each $i \in [2]$ and $Y \subseteq X \subseteq V(R)$, we say that a generalised clique Z_t in R on (X,Y) is *of colour i* if $\phi^{\text{ind}}(k) = \phi^{\text{ind}}(pq) = i$, for all $k \in Y$ and $pq \in \binom{X}{2}$. We say that R is *(Z_{t_1}, Z_{t_2})-free* if there is no Z_{t_i} of colour i for any $i \in [2]$. It was implicitly proven in the proof Theorem 1.3 in [2] that a Z_t of colour i in R implies $K_t \subseteq G_i$. This implies the following, since G is (K_3, K_4)-free:

$$R \text{ is } (Z_3, Z_4)\text{-free}. \tag{5.5}$$

Let $U_1 \cup U_2 \cup U_3$ be a max-cut 3-partition of R. The desired partition of $V(G)$ will be an adjustment of this partition. By (5.1) and the definition of max-cut and Theorem 2.3, it is easy to see that we have

$$\sum_{i\in[3]} e(R[U_i]) \leq \gamma^{1/3}m^2, \qquad |U_i| = m/3 \pm \gamma^{1/7}m \text{ and} \tag{5.6}$$

$$\delta^{\text{cr}}(R[U_1, U_2, U_3]) \geq (\delta(R) - \max_{i\in[3]} |U_i|)/2 \geq m/7. \tag{5.7}$$

We will obtain the colour pattern of R in ϕ^{ind}. First we show that each vertex set U_i is monochromatic in ϕ^{ind}.

Claim 5.4. *For every $i \in [3]$, there exist $j \in [2]$ such that $\phi^{\text{ind}}(U_i) = j$. In particular, we have*

$$\alpha(G_j[\cup_{k\in U_i} V_k]) \leq \sqrt{\delta}n.$$

Proof. Suppose the lemma is not true, then by symmetry, we may assume that $\phi(U_1) \neq j$ for any $j \in [2]$. Let $W := \{w \in U_1 : \phi^{\text{ind}}(w) = 2\}$. We shall argue that one of the following two cases must happen and then derive contradictions in each case.

Case 1. There exists vertices $u, w \in U_1$ $v_2 \in U_2$, $v_3 \in U_3$ such that $\{v_2v_3, uv_2, uv_3\} \subseteq E(R)$ and $wv_2, wv_3 \subseteq E(R')$ and $\phi^{\text{ind}}(u) = 1, \phi^{\text{ind}}(w) = 2$.

Case 2. There exists vertices $u, w \in U_1$ $v_2 \in U_2$, $v_3 \in U_3$ such that $R[\{u, w, v_2, v_3\}]$ induces a copy of K_4 and $\phi^{\text{ind}}(u) = 1, \phi^{\text{ind}}(w) = 2$.

Suppose that $|W| \geq m/100$. Fix an arbitrary $u \in U_1$ with $\phi^{\text{ind}}(u) = 1$. Then, by (5.4), more than half of the vertices w in W satisfy $d_{R'}(w) \geq d_R(w) - \gamma^{1/5}m$, and as $\sum_{i\in[3]} e(R[U_i]) \leq \gamma^{1/3}m^2$, more than half of the vertices w in W satisfies $|N_R(w, U_1)| \leq \gamma^{1/4}m$. Hence there exists $w \in W$ with $|N_{R'}(w, U_i)| \geq \delta(R) - |U_{5-i}| - 2\gamma^{1/5}m \geq m/4$ for each $i \in \{2, 3\}$. By this and (5.7), for each $i \in \{2, 3\}$ we have

$$|N_R(u, U_i) \cap N_{R'}(w, U_i)| \geq m/7 + m/4 - |U_i| \geq m/30.$$

Together with (5.1) and the definition of max-cut partition, this implies that there exists an edge v_2v_3 between $N_R(u, U_2) \cap N_{R'}(w, U_2)$ and $N_R(u, U_3) \cap N_{R'}(w, U_3)$, yielding Case 1.

We may then assume that $|W| \leq m/100$. Fix an arbitrary $w \in W$. If $|N_R(w, U_1)| > m/50$, then we have $|N_R(w, U_1 \setminus W)| \geq m/100$. As $\sum_{i\in[3]} e(R[U_i]) \leq \gamma^{1/3}m^2$, more than half of the vertices u in $N_R(w, U_1 \setminus W)$ satisfy $|N_R(u, U_1)| \leq \gamma^{1/4}m$. Hence there exists $u \in N_R(w, U_1 \setminus W)$ with $|N_R(u, U_i)| \geq m/4$ for each $i \in \{2, 3\}$. By this and (5.7), for each $i \in \{2, 3\}$ we have

$$|N_R(u, U_i) \cap N_R(w, U_i)| \geq m/7 + m/4 - |U_i| \geq m/30.$$

Thus by (5.1) and the definition of max-cut partition, there exists an edge v_2v_3 between $N_R(uw, U_2)$ and $N_R(uw, U_3)$, yielding Case 2.

Thus we may assume that $|N_R(w, U_1)| \leq m/50$, thus $d_R(w) \leq |U_2|+|U_3|+m/50 \leq (2/3+1/40)m$. Together with (5.3), this implies that

$$d_{R'}(w) \geq 4m/3 - (2/3 + 1/40)m - 9\gamma m \geq (2/3 - 1/30)m.$$

Hence, for each $i \in \{2, 3\}$, we have

$$|N_{R'}(w, U_i)| \geq d_{R'}(w) - |N_R(w, U_1)| - |U_{5-i}| \geq (1/3 - 1/30 - 1/40)m \geq m/4.$$

By (5.7), there exists a vertex $u \in U_1 \setminus W$ such that for each $i \in \{2, 3\}$, we have

$$|N_R(u, U_i) \cap N_{R'}(w, U_i)| \geq m/4 + m/7 - |U_2| \geq m/30.$$

Thus by (5.1) and the definition of max-cut partition, there exists an edge v_2v_3 between $N_R(u, U_2) \cap N_{R'}(w, U_2)$ and $N_R(u, U_3) \cap N_{R'}(w, U_3)$, yielding again Case 1.

For each $i \in \{2, 3\}$ and $j \in [2]$, if $\phi^{\text{ind}}(U_i) = j$, then, by the definition of ϕ^{ind}, we have $\cup_{k\in U_i} V_k \subseteq V_j^*$, and so $\alpha(G_j[\cup_{k\in U_i} V_k]) \leq \alpha(G_j[V_j^*]) \leq \sqrt{\delta}n$ as desired. We shall now derive contradictions in each case to finish the proof.

Suppose **Case 1** happens. By the definition of R', for each $i \in \{2, 3\}$ we have $d(wv_i) \geq 1/2 + \gamma$. As $\phi^{\text{ind}}(u) = 1$, we must have $\phi^{\text{ind}}(uv_i) = 2$ for $i \in \{2, 3\}$, otherwise we get a Z_3 of colour 1 on (uv_i, u), contradicting (5.5). Suppose now that $\phi^{\text{ind}}(v_2v_3) = 2$. For each $i \in \{2, 3\}$, it must be that $\phi^{\text{ind}}(v_i) = 1$, otherwise (uv_2v_3, v_i) is a Z_4 of colour 2, which in turn implies that $\phi^{\text{ind}}(wv_i) = 2$, otherwise (wv_i, v_i) is a Z_3 of colour 1. But then (wv_2v_3, w) is a Z_4 of colour 2, a contradiction.

Hence, we may assume that $\phi^{\text{ind}}(v_2v_3) = 1$. For each $i \in \{2, 3\}$, we must have $\phi^{\text{ind}}(v_i) = 2$, otherwise we get a Z_3 of colour 1 on (v_2v_3, v_i), a contradiction. As $d(wv_i) \geq 1/2+\gamma$ and $\phi^{\text{ind}}(w) = 2$, we must have $\phi^{\text{ind}}(wv_i) = 1$, otherwise we get a Z_4 of colour 2 on (wv_i, wv_i). However, then we have a Z_3 of colour 1 on $(wv_2v_3, \emptyset)$, a contradiction.

Suppose **Case 2** happens. As $\phi^{\text{ind}}(u) = 1$, we must have $\phi^{\text{ind}}(uw) = \phi^{\text{ind}}(uv_i) = 2$ for $i \in \{2, 3\}$, otherwise we get a Z_3 of colour 1 on (uv_i, u) or (uw, u), contradicting (5.5).

Suppose now that $\phi^{\text{ind}}(v_2v_3) = 2$. Then for each $i \in \{2, 3\}$, we have $\phi^{\text{ind}}(v_i) = 1$, otherwise (uv_2v_3, v_i) is a Z_4 of colour 2, which in turn implies that $\phi^{\text{ind}}(wv_i) = 2$ for each $i \in \{2, 3\}$. But then $(wuv_2v_3, \emptyset)$ is a Z_4 of colour 2, a contradiction.

Hence, we may assume that $\phi^{\text{ind}}(v_2v_3) = 1$, Then for each $i \in \{2, 3\}$, we must have $\phi^{\text{ind}}(v_i) = 2$, otherwise we get Z_3 of colour 1 on (v_2v_3, v_i). Moreover, for each $i \in \{2, 3\}$, we must have $\phi^{\text{ind}}(wv_i) = 1$, otherwise (v_iuw, w) is a Z_4 of colour 2. But then $(wv_2v_2, \emptyset)$ forms a Z_3 of colour 1, a contradiction. □

Claim 5.5. *By permuting indices of U_1, U_2, U_3, we may assume the following. We have $\phi^{\mathrm{ind}}(U_1) = 1$ and for each $i \in \{2,3\}$, we have $\phi^{\mathrm{ind}}(U_1, U_i) = 2$ and one of the following holds.*

(B1) $\phi^{\mathrm{ind}}(U_2) = \phi^{\mathrm{ind}}(U_3) = 2$ *and* $\phi^{\mathrm{ind}}(U_2, U_3) = 1$*; or*
(B2) $\phi^{\mathrm{ind}}(U_2) = \phi^{\mathrm{ind}}(U_3) = 1$ *and* $\phi^{\mathrm{ind}}(U_2, U_3) = 2$.

Proof. If $\phi^{\mathrm{ind}}(U_i) = 2$ for all $i \in [3]$, then it is easy to see that all crossing edges of R' are of colour 1, otherwise we obtain a generalised clique Z_4 of colour 2. However, then we can easily check that R contains a copy of K_3 of colour 1, which is again a contradiction.

Hence, by Claim 5.4, we may assume that $\phi^{\mathrm{ind}}(U_1) = 1$. Then as R does not have a generalised clique Z_3 of colour 1, we have that $\phi^{\mathrm{ind}}(U_1, U_i) = 2$ for $i \in \{2,3\}$. If $\phi^{\mathrm{ind}}(U_2) = 2$, then $\phi^{\mathrm{ind}}(U_2, U_3) = 1$, otherwise we get a generalised clique Z_4 of colour 2. But then we must have $\phi^{\mathrm{ind}}(U_3) = 2$, giving (B1). Similarly if $\phi^{\mathrm{ind}}(U_2) = 1$, we obtain (B2). □

Let $X'_i := \cup_{k \in U_i} V_k$ for each $i \in [3]$ and further add V_0 to X'_1. Then $V(G) = X'_1 \cup X'_2 \cup X'_3$. Note that (5.6) implies that for each $i \in [3]$ we have $|X'_i| = \frac{n}{3} \pm \frac{3}{2}\gamma^{1/7}n$. Then we have

$$\sum_{i\in[3]} e(G[X'_i]) \leq \sum_{i\in[3]} e(R[U_i]) \cdot \left(\frac{n}{m}\right)^2 + \varepsilon n^2 + \frac{n^2}{m} + 2\gamma n^2 \overset{(5.6)}{\leq} 2\gamma^{1/3}n^2. \tag{5.8}$$

Note that (5.7) provides a minimum crossing degree of R with respect to the partition $U_1 \cup U_2 \cup U_3$. However, in G, some vertex could have low crossing degree with respect to the partition $X'_1 \cup X'_2 \cup X'_3$. To amend this problem, we will consider the following modified partition $X_1 \cup X_2 \cup X_3$ of $V(G)$.

Claim 5.6. *There exists a partition $X_1 \cup X_2 \cup X_3$ of $V(G)$ such that the following holds.*

(X1) *For each $i \in [3]$, we have $|X_i| = n/3 \pm 2\gamma^{1/7}n$ and $|X_i \triangle X'_i| \leq 20\gamma^{1/3}n$.*
(X2) $\delta^{\mathrm{cr}}(G[X_1, X_2, X_3]) \geq n/10$.

Proof. For all $i \in [3]$ and $v \in X'_i$, if $d(v, X'_j) \leq n/10$ for some $j \neq i$, then move v to X'_j. We repeat this until no such vertex exists. Let the resulting set be X_i, $i \in [3]$. We first show that this process terminates and so X_is are well-defined.

Recall that $\delta(G) \geq 2n/3$, so if there exist $ij \in \binom{[3]}{2}$ and $v \in X'_i$ with $d(v, X'_j) \leq n/10$, we see that for each $k \neq j$,

$$d(v, X'_k) \geq \delta(G) - n/10 - \max_{i\in[3]} |X'_i| \geq n/5.$$

Thus, after moving v from X'_i to X'_j, the number of inner edges decreases by at least $n/5 - n/10 = n/10$. Hence, by (5.8), after moving at most $2\gamma^{1/3}n^2/(n/10) = 20\gamma^{1/3}n$ vertices, the process stops. Hence, we obtain (X1) proving the first part and (X2) holds by definition. □

Note that (A1) holds due to (X1). By Claims 5.4, 5.5 and (X1), we have (A2) as

$$\alpha(G_1[X_1]) \leq \alpha(G_1[X'_1]) + ||X'_1| - |X_1|| \leq \sqrt{\delta}n + 20\gamma^{1/3}n \leq \gamma^{1/4}n. \tag{5.9}$$

For what follows, we assume (B1) holds, which then leads to (P1) ((B2) implying (P2) can be proven analogously). Similar to (5.9), (B1) implies that $\alpha(G_2[X_i]) \leq \gamma^{1/4}n$ for $i \in \{2,3\}$, proving the first part of (P1).

We now bound $\Delta(G[X_i])$ for each $i \in \{2,3\}$. Without loss of generality, it is enough to bound $\Delta(G[X_2])$. Note first that, as G_1 is K_3-free, by (5.9), for each $v \in V(G)$, we have

$$d_{G_1}(v, X_1) \leq \alpha(G_1[X_1]) \leq \gamma^{1/4}n. \tag{5.10}$$

Define

$$J := \bigcup_{i\in[3]} \{v \in X_i : \ d(v, \overline{X_i}) \leq |\overline{X_i}| - \gamma^{1/8}n\}$$

to be the set of vertices with large missing crossing degree. By Claim 5.6 and (5.8), we have

$$\sum_{i\in[3]} e(G[X_i]) \le \sum_{i\in[3]} (e(G[X_i']) + ||X_i'| - |X_i||n) \le \gamma^{1/4}n^2,$$

and so, as $e(G) \ge \frac{n^2}{3}$ and $e(K_{|X_1|,|X_2|,|X_3|}) \le n^2/3$, we have that

$$e(\overline{G}[X_1, X_2, X_3]) \le \frac{n^2}{3} - (e(G) - \sum_{i\in[3]} e(G[X_i])) \le \sum_{i\in[3]} e(G[X_i]) \le \gamma^{1/4}n^2 \text{ and}$$

$$|J| \le \frac{2e(\overline{G}[X_1, X_2, X_3])}{\gamma^{1/8}n} = 2\gamma^{1/8}n. \tag{5.11}$$

We claim that for each $y \in X_3$, we have

$$d_{G_2}(y, X_2) \le 3\gamma^{1/8}n \text{ and } d_{G_1}(y, X_3) \le |J| \le 2\gamma^{1/8}n. \tag{5.12}$$

Indeed, suppose that $d_{G_2}(y, X_2) > 3\gamma^{1/8}n$. Then $|N_{G_2}(y, X_2) \setminus J| \ge \gamma^{1/8}n > \alpha(G_2[X_2])$, and so there exists $uv \in E(G_2)$ with $u, v \in N_{G_2}(y, X_2) \setminus J$. By (X2), (5.10) and the definition of J, we have

$$|N_{G_2}(\{u, v, y\}, X_1)| \ge \delta^{\mathrm{cr}}(G[X_1, X_2, X_3]) - \sum_{x\in\{u,v,y\}} d_{G_1}(x, X_1) - 2\cdot\gamma^{1/8}n \ge n/20,$$

showing that $K_4 \subseteq G_2$, a contradiction, thus the first part of (5.12) holds.

Suppose that $d_{G_1}(y, X_3) > |J|$. So there exists $yy' \in E(G_1[X_3])$ with $y' \notin J$. But then the first part of (5.12) implies that

$$|N_{G_1}(\{y, y'\}, X_2)| \ge \delta^{\mathrm{cr}}(G[X_1, X_2, X_3]) - \sum_{x\in\{y,y'\}} d_{G_2}(x, X_2) - \gamma^{1/8}n \ge n/20,$$

contradicting $K_3 \not\subseteq G_1$. Thus (5.12) holds.

We now show that for each $y \in X_3$, we have $d_{G_2}(y, X_3) \le 3\gamma^{1/17}n$, which together with (5.12) implies that $\Delta(G[X_3]) \le \gamma^{1/18}n$. Fix an arbitrary $y \in X_3$ and let $Y := N_{G_2}(y, X_3)$. suppose to the contrary that $|Y| > 3\gamma^{1/17}n$. For $i \in [2]$, define

$$J_i := \{v \in X_i : d_{G_i}(v, X_3) \ge \gamma^{1/16}n\}.$$

By (5.10) and (5.12), we get, for each $i \in [2]$, that

$$|J_i| \le \frac{e(G_i[X_i, X_3])}{\gamma^{1/16}n} \le \frac{|X_3|\cdot\max\{\gamma^{1/4}n, 3\gamma^{1/8}n\}}{\gamma^{1/16}n} \le 3\gamma^{1/16}n. \tag{5.13}$$

As $|N_{G_2}(y, X_1)| \ge \delta^{\mathrm{cr}}(G[X_1, X_2, X_3]) - d_{G_1}(y, X_1) > |J| + |J_1|$ due to (5.10) and (5.11), we can pick $u \in N_{G_2}(y, X_1) \setminus (J \cup J_1)$. By the definition of J and J_1, we have

$$d_{G_2}(u, Y) \ge |Y| - d_{\overline{G}}(u, \overline{X_1}) - d_{G_1}(u, X_3) \ge |Y| - \gamma^{1/17}n.$$

Similarly, we can pick $v \in X_2 \setminus (J \cup J_2)$ with $d_{G_1}(v, Y) \ge |Y| - \gamma^{1/17}n$. Thus, writing $Y' := N_{G_1}(v, Y) \cap N_{G_2}(u, Y)$, we have $|Y'| \ge |Y| - 2\gamma^{1/17}n > \alpha(G)$. So there exists $xx' \in E(G[Y'])$. However, if $\phi(xx') = 1$, then $\{x, x', v\}$ induces a K_3 in G_1; while if $\phi(xx') = 2$, then $\{x, x', u, y\}$ induces a K_4 in G_2, a contradiction. This shows $\Delta(G[X_3]) \le \gamma^{1/18}n$, and $\Delta(G[X_2]) \le \gamma^{1/18}n$.

To bound $\Delta(G[X_1])$, we need to first prove (A4) that no vertex in X_1 can have high G_1-degree to both X_2 and X_3. Suppose that $v \in X_1$ is such that $d_{G_1}(v, X_i) \ge \gamma^{1/9}n > |J|$ for both $i \in \{2, 3\}$. Fix an arbitrary $u \in N_{G_1}(v, X_3) \setminus J$. Then by (5.12) and the fact that $u \notin J$, we have

$$|N_{G_1}(\{v, u\}, X_2)| \ge d_{G_1}(v, X_2) - d_{\overline{G}}(u, \overline{X_3}) - d_{G_2}(u, X_2) \ge \gamma^{1/8}n,$$

which contradicts $K_3 \not\subseteq G_1$, proving (A4).

Fix an arbitrary $w \in X_1$, suppose to the contrary that $d(w, X_1) \geq \gamma^{1/18} n > d_{G_1}(w, X_1) + |J \cup J_1|$, due to (5.10) and (5.11) and (5.13). Fix a vertex $u \in N_{G_2}(w, X_1) \setminus (J \cup J_1)$. By (A4), we may assume that $d_{G_1}(w, X_3) < \gamma^{1/9} n$. Then by (X2) and the fact that $u \notin J \cup J_1$, we have

$$|N_{G_2}(\{w,u\}, X_3)| \geq \delta^{\rm cr}(G[X_1, X_2, X_3]) - d_{G_1}(w, X_3) - d_{\overline{G}}(u, \overline{X_1}) - d_{G_1}(u, X_3) > \alpha(G_2[X_3]),$$

contradicting $K_4 \not\subseteq G_2$. Thus, for each $i \in [3]$, we have $\Delta(G[X_i]) \leq \gamma^{1/18} n$, proving (A3). Consequently,

$$\delta^{\rm cr}(G[X_1, X_2, X_3]) \geq \delta(G) - \max_{i \in [3]} (\Delta(G[X_i]) + |X_i|) \geq n/3 - \gamma^{1/19} n,$$

proving (A5). Together with (A1), this implies that for all $i \in \{2, 3\}$ and $v \in X_i$, we have

$$d_{G_2}(v, X_1) \geq \delta^{\rm cr}(G[X_1, X_2, X_3]) - d_{G_1}(v, X_1) \overset{(5.10)}{\geq} |X_1| - \gamma^{1/20} n,$$

and that

$$d_{G_1}(v, X_{5-i}) \geq \delta^{\rm cr}(G[X_1, X_2, X_3]) - d_{G_2}(v, X_{5-i}) \overset{(5.12)}{\geq} |X_{5-i}| - \gamma^{1/20} n,$$

proving (A6) and the second part of (P1) as desired. □

5.2. **Proof of Lemma 5.1.** Suppose that $e(G) > (\frac{1}{3} + \frac{\delta}{2} + \frac{3\delta^2}{2}) n^2$. By applying Lemma 2.9 with $G, 2/3, \delta + 3\delta^2$ playing the roles of G, d, ε, respectively, to obtain an n'-vertex graph G' with $n' \geq \delta^{1/2} n/2$. Then $\delta(G') \geq 2n'/3$. Let $\delta' := \delta n/n' \in [\delta, \delta^{1/3}]$. Since $1 \leq \alpha(G') \leq \alpha(G) = \delta n = \delta' n'$, we have

$$e(G') \geq \frac{n'^2}{3} + \left(\frac{\delta}{2} + \frac{3\delta^2}{2}\right) n^2 - \frac{n - n'}{3} \geq \left(\frac{1}{3} + \frac{\delta'}{2} + \frac{3\delta'^2}{2}\right) n'^2. \tag{5.14}$$

Note that ϕ still induces an edge-colouring of G' which is (K_3, K_4)-free. As $1/n \ll \delta \ll \gamma$ and $n' \geq \delta^{1/2} n/2$ and $\delta' \in [\delta, \delta^{1/3}]$, we can apply Lemma 5.2 with G', δ', γ playing the roles of G, δ, γ to obtain a partition $X_1 \cup X_2 \cup X_3$ of $V(G')$ satisfying (A1)–(A6). We assume that (P1) occurs.[3]. Define

$$A := \{v \in X_1 : d_{G'_1}(v, X_2) \geq n'/5\} \quad \text{and} \quad B := \{v \in X_1 : d_{G'_1}(v, X_3) \geq n'/5\}.$$

Note that (A4) implies that $A \cap B = \emptyset$. Bounding $e(G)$ amounts to show the following claim.

Claim 5.7. *The following hold:*

- (G′1) *For each $i \in \{2, 3\}$, the graph $G'[X_i]$ is K_3-free.*
- (G′2) *Both A and B are independent sets and so $|A|, |B| \leq \alpha(G') \leq \delta' n'$.*
- (G′3) *Both $G'[X_1 \setminus A]$ and $G'[X_1 \setminus B]$ are K_3-free.*

First, we show how Claim 5.7 implies Lemma 5.1. For each $i \in \{2, 3\}$, (G′1) implies that $\Delta(G'[X_i]) \leq \alpha(G') \leq \delta' n'$, and so $e(G'[X_i]) \leq \delta' n' |X_i|/2$. On the other hand, (G′2) and (G′3) imply that $\Delta(G'[X_1 \setminus A]), \Delta(G'[X_1 \setminus B]) \leq \alpha(G') \leq \delta' n'$, and so $e(G'[A, X_1 \setminus (A \cup B)]) \leq \delta' n' |A|$. Therefore,

$$\begin{aligned} e(G'[X_1]) &= e(G'[X_1 \setminus A]) + e(G'[A, B]) + e(G'[A, X_1 \setminus (A \cup B)]) \\ &\leq (|X_1| - |A|) \delta' n'/2 + |A| \delta' n' + \delta' n' |A| \leq \delta' n' |X_1|/2 + 3\delta'^2 n'^2/2. \end{aligned}$$

Thus, we have

$$e(G') \leq e(G'[X_1, X_2, X_3]) + \sum_{i \in [3]} e(G'[X_i]) \leq n'^2/3 + \delta' n'^2/2 + 3\delta'^2 n'^2/2,$$

contradicting (5.14). Thus we conclude that $e(G) \leq (\frac{1}{3} + \frac{\delta}{2} + \frac{3\delta^2}{2}) n^2$.

---

[3]The (P2) case is only easier, we include its proof in the online arXiv version. In fact, graphs satisfying (P2) case can only have at most $n'^2/3 + \delta n'^2/2$ edges, a contradiction

Proof of Claim 5.7. Fix arbitrary $i \in \{2,3\}$. Suppose that $T = \{u,v,w\}$ induces a triangle in $G'[X_i]$. By (A1) and (P1), we see that $|N_{G'_1}(T, X_{5-i})| \geq n/4$. As G'_1 is K_3-free, this implies that T is monochromatic in colour 2. But then, by (A6), we have $|N_{G'_2}(T, X_1)| \geq n/4$. This contradicts $K_4 \not\subseteq G'_2$, proving (G′1).

Suppose that uv is an edge in $G'[A]$ (the proof for B is similar). By the definition of A and (A1), we have $|N_{G'_1}(uv, X_2)| \geq 2n/5 - |X_2| \geq n/20$, implying that $\phi(uv) = 2$ as G'_1 is K_3-free. Also by (A4) and the fact that $u, v \in A$, we see that both $d_{G'_1}(u, X_3)$ and $d_{G'_1}(v, X_3)$ are less than $\gamma^{1/9}n$. Thus, by (A5), we have that

$$|N_{G'_2}(uv, X_3)| \geq 2(\delta^{\mathrm{cr}}(G'[X_1, X_2, X_3]) - \gamma^{1/9}n) - |X_3| \geq n/4 > \alpha(G'_2[X_3]).$$

Hence, there exists an edge of colour 2 in $N_{G'_2}(uv, X_3)$, contradicting $K_4 \not\subseteq G'_2$, proving (G′2).

Suppose $T = \{u,v,w\}$ induces a triangle in $X_1 \setminus B$ (the proof for $X_1 \setminus A$ is similar). Since G'_1 is K_3-free, we may assume that $\phi(uw) = 2$. To prove (G′3), it suffices to show that $|N_{G'_2}(uw, X_i)| \geq n/30 > \alpha(G'_2[X_i])$ for some $i \in \{2,3\}$, since then $K_4 \subseteq G'_2$, a contradiction. As A is an independent set due to (G′2), we may further assume that $w \notin A \cup B$. By the definition of A and B, we have

$$d_{G'_2}(w, X_i) \geq \delta^{\mathrm{cr}}(G'[X_1, X_2, X_3]) - n/5 \geq n/10, \quad \forall\, i \in \{2,3\}. \tag{5.15}$$

For each $i \in \{2,3\}$, let $W_i := N_{G'_2}(w, X_i)$. Note that $d_{G'_1}(u, X_2) \geq d_{G'_1}(u, W_2) \geq n/20$, since otherwise $|N_{G'_2}(uw, X_2)| \geq n/30$ as desired. Then (A4) implies that $d_{G'_1}(u, X_3) < \gamma^{1/9}n$. Together with (A1), (A5) and (5.15), we have

$$|N_{G'_2}(uw, X_3)| \geq \delta^{\mathrm{cr}}(G'[X_1, X_2, X_3]) - d_{G'_1}(u, X_3) + d_{G'_2}(w, X_3) - |X_3| \geq n/30,$$

as desired. □

This completes the proof of Lemma 5.1, providing the second equality of Theorem 1.1.

6. Stability for $\varrho(K_3, K_3, \delta)$ without regularity

In this section, we present the upper bound on $\varrho(K_3, K_3, \delta)$.[4] For convenience, we rephrase the upper bound as follows.

Lemma 6.1. *Suppose $0 < 1/n \ll \delta < 10^{-13}$. Let G be an n-vertex (K_3, K_3)-free graph with $\alpha(G) \leq \delta n$. Then*

$$e(G) \leq \frac{n^2}{4} + \frac{\delta n^2}{2}.$$

We will prove Lemma 6.1 using the following coloured stability.

Lemma 6.2. *Suppose $0 < 1/n \ll \delta < 10^{-6}$. Let G be an n-vertex (K_3, K_3)-free graph with $\alpha(G) \leq \delta n$ and $\delta(G) \geq n/2$. Then for any (K_3, K_3)-free $\phi : E(G) \to [2]$, there exists a partition $V(G) = A \cup B$ with $|A|, |B| = n/2 \pm \delta^{1/3}n$ and $i \in [2]$ such that $\delta(G_i[A, B]) \geq 2n/5$.*

We will present a proof of Lemma 6.2 without regularity lemma. First, we show how it implies Lemma 6.1.

Proof of Lemma 6.1. Suppose that $e(G) > (\frac{1}{4} + \frac{\delta}{2})n^2$. By applying Lemma 2.9 with $G, 1/2, \delta$ playing the roles of G, d, ε, respectively, to obtain an n'-vertex graph G' with $n' \geq \delta^{1/2}n/2$ satisfying the following, where $\delta' := \delta n/n' \in [\delta, 10^6)$:

$$\delta(G') \geq n'/2 \ \text{ and } \ e(G') \geq \frac{n'^2}{4} + \frac{\delta n^2}{2} - \frac{n - n'}{4} \geq \left(\frac{1}{4} + \frac{\delta'}{2}\right) n'^2.$$

Moreover, $\alpha(G') \leq \alpha(G) = \delta n = \delta' n'$.

[4]The upper bound on $\varrho(K_3, K_5, \delta)$ can be proved by combining ideas in the proofs of the upper bounds on $\varrho(K_3, K_3, \delta)$ and $\varrho(K_3, K_4, \delta)$, we include its proof in the online arXiv version.

As G is (K_3, K_3)-free, $G' \subseteq G$ is also (K_3, K_3)-free and there exists an (K_3, K_3)-free 2 edge-colouring ϕ of G'. As $1/n \ll \delta$ and $n' \geq \delta^{1/2}n/2$ and $\delta' < 10^{-6}$, we apply Lemma 6.2 to obtain a partition $A \cup B$ of $V(G')$ with $|A|, |B| = n'/2 \pm \delta'^{1/3}n$ and $\delta(G'_1[A, B]) \geq 2n'/5$.

We claim that both $G'[A]$ and $G'[B]$ are triangle-free. Suppose that $T \in \binom{A}{3}$ induces a triangle in $G'[A]$. Since G'_1 is K_3-free and $|N_{G'_1}(T, B)| \geq 3\delta(G'_1[A, B]) - 2|B| > n/6$, we see that no edge in T can be of colour 1. But then T is monochromatic in colour 2, contradicting $K_3 \not\subseteq G'_2$. Thus, $e(G'[A]) \leq \Delta(G'[A])|A|/2 \leq \alpha(G')|A|/2 \leq \delta' n'|A|/2$. Similarly, $e(G'[B]) \leq \delta' n'|B|/2$. Hence,

$$e(G') = e(G'[A, B]) + e(G'[A]) + e(G'[B]) \leq n'^2/4 + \delta' n'^2/2,$$

a contradiction. Thus we conclude $e(G) \leq (\frac{1}{4} + \frac{\delta}{2})n^2$. □

Proof of Lemma 6.2. Assume without loss of generality that $e(G_1) \geq e(G_2)$, so $e(G_1) \geq n^2/8$. It is easy to see that the following fact follows from G being (K_3, K_3)-free and from $\alpha(G) \leq \delta n$.

Fact 6.3. *For all $x, y \in V(G)$, we have $|N_{G_1}(x) \cap N_{G_2}(y)| \leq \alpha(G) \leq \delta n$.*

We will sequentially choose four vertices as follows.

- Take a vertex x with maximum G_1-degree, i.e. $d_{G_1}(x) = \Delta(G_1)$, and set $X := N_{G_1}(x)$. Then $|X| \geq \frac{2e(G_1)}{n} \geq n/4$.
- Choose a vertex $y \in X$ with maximum G_1-degree and set $Y := N_{G_1}(y)$. Note that as G_1 is K_3-free, $X \cap Y = \emptyset$. Denote $Z := V(G) \setminus (X \cup Y)$ and $\alpha := |Z|/n$.
- Pick now $x' \in Z$ with maximum G_2-degree in Z. Let $X' := N_{G_2}(x', Z)$ and $\beta := |X'|/n$. By definition, we have $0 \leq \beta \leq \alpha$.
- Finally, take $y' \in X'$ with maximum G_2-degree in Z and set $Y' := N_{G_2}(y', Z)$. Similarly, as G_2 is K_3-free, $X' \cap Y' = \emptyset$. So $|Y'| \leq |Z \setminus X'| = (\alpha - \beta)n$.

Claim 6.4. *We have $|X| + |Y| \geq n/3$, consequently, $\alpha \leq 2/3$.*

Proof. Let $a := |X|/n$ and $b := |Y|/n$. By the definition of x and X, every vertex in X (resp. not in X) has G_1-degree at most $|Y|$ (resp. $|X|$). Thus,

$$\frac{n^2}{4} \leq 2e(G_1) \leq \sum_{u \notin X} d_{G_1}(u) + \sum_{v \in X} d_{G_1}(v) \leq (n - |X|)|X| + |X||Y|. \tag{6.1}$$

We then have

$$16/9 < 2 \leq 4 \cdot 2a(1 - a + b) \leq (2a + (1 - a + b))^2,$$

whence $4/3 < 1 + a + b$, i.e. $\alpha = 1 - a - b < 2/3$ as desired. □

Let us show the following bound on the size of G_1:

$$e(G_1) \leq \frac{(1 - \beta)^2 n^2}{4} + \delta n^2. \tag{6.2}$$

Indeed, the first term above bounds $e(G_1[\overline{X'}])$ as G_1 is K_3-free; while the second term bounds all G_1-edges with at least one endpoints in X'. To see this, for each vertex $v \in V(G)$, we have $d_{G_2}(v, X') \subseteq N_{G_1}(v) \cap N_{G_2}(x')$, thus the desired bound follows from Fact 6.3. Similarly, we can bound all G_2-edges with at least one endpoints in $X \cup Y$ by $2\delta n^2$. Thus, we have that $e(G_2) \leq e(G_2[Z]) + 2\delta n^2$.

Claim 6.5. *We have that $e(G_2) \leq 22\delta n^2$.*

Proof. As observed above, $e(G_2) \leq e(G_2[Z]) + 2\delta n^2$. Note also that, by the definition of x',

$$e(G_2[Z]) \leq \frac{|X'| \cdot |Z|}{2} = \frac{\alpha\beta n^2}{2}. \tag{6.3}$$

On the other hand, analogous to (6.1), by the definition of y', we have

$$\begin{aligned}
e(G_2[Z]) &\le \frac{1}{2}\left(\sum_{u\in Z\setminus X'} d_{G_2}(u,Z)+\sum_{v\in X'} d_{G_2}(v,Z)\right)\\
&\le \frac{1}{2}\left(|Z\setminus X'|\cdot\beta n+|X'|\cdot(\alpha-\beta)n\right)=(\alpha-\beta)\beta n. \qquad (6.4)
\end{aligned}$$

By (6.2) and (6.3), we have $\frac{n^2}{4}\le e(G)\le\frac{n^2}{4}((1-\beta)^2+2\alpha\beta)+3\delta n^2$, and so

$$3(\beta^2-2\beta+2\alpha\beta+12\delta)\ge 0; \qquad (6.5)$$

and by (6.2) and (6.4), we obtain $\frac{n^2}{4}\le e(G)\le\frac{n^2}{4}((1-\beta)^2+4(\alpha-\beta)\beta)+3\delta n^2$, therefore

$$4\alpha\beta-2\beta-3\beta^2+12\delta\ge 0. \qquad (6.6)$$

Now summing (6.5) with (6.6), we get $10\alpha\beta-8\beta+48\delta\ge 0$. Recall that $\alpha\le 2/3$, we then have $12\alpha\beta\le 8\beta\le 10\alpha\beta+48\delta$, implying that $\alpha\beta\le 24\delta$. Thus

$$e(G_2)\le e(G_2[Z])+2\delta n^2\le\alpha\beta n^2/2+2\delta n^2\le 14\delta n^2,$$

as desired. □

By Claim 6.5, we have $e(G_1)\ge n^2/4-22\delta n^2$. Apply Theorem 2.3 to G_1 with $t=22\delta n^2$ and let $V(G)=V(G_1)=A\cup B$ be an arbitrary max-cut partition of G_1. Then we have

$$e(G[A])+e(G[B])\le 3t=66\delta n^2\ \Rightarrow\ e(G_1[A,B])\ge e(G_1)-66\delta n^2\ge\frac{n^2}{4}-88\delta n^2, \qquad (6.7)$$

and $|A|,|B|=n/2\pm 2\sqrt{t}=n/2\pm 10\sqrt{\delta}n$. Note that there exists a vertex $v\in B$ with

$$d_{G_1}(v,A)\ge\frac{e(G_1[A,B])}{|B|}\ge\frac{n^2/4-88\delta n^2}{n/2+10\sqrt{\delta}n}\ge\frac{n}{2}-10\sqrt{\delta}n\ge|A|-20\sqrt{\delta}n.$$

Consequently, for any $u\in V(G)$, we have $d_{G_2}(u,A)\le 21\sqrt{\delta}n$, as otherwise $|N_{G_2}(u)\cap N_{G_1}(v)|\ge\sqrt{\delta}n$, contradicting Fact 6.3. Similarly, $d_{G_2}(u,B)\le 21\sqrt{\delta}n$. Thus we have $\Delta(G_2)\le 42\sqrt{\delta}n$.

We claim that $A\cup B$ is the desired partition. Suppose $d_{G_1}(w,B)<2n/5$ for some $w\in A$. Then $d_{G_1}(w,A)\ge\delta(G)-\Delta(G_2)-d_{G_1}(w,B)\ge n/20$. As $A\cup B$ is a max-cut, we see that $d_{G_1}(w,B)\ge d_{G_1}(w,A)-\Delta(G_2)\ge n/30$. Since G_1 is K_3-free, there is no edge of G_1 in $[N_{G_1}(w,A),N_{G_1}(w,B)]$, implying that $e(G_1[A,B])\le n^2/4-(n/20)\cdot(n/30)$, contradicting (6.7) and that $\delta<10^{-6}$. □

7. Proof of Theorem 1.2

7.1. **Upper bound.** Let $s\ge 2$ and fix a function $g_s(n)$ satisfying (1.2). Note that a function $g'_s(n)$ satisfying $g_s(n)=(g'_s(n)/n)^2n$ is also a function satisfying (1.2). We choose constants such that $0<1/n\ll 1/M'\ll\varepsilon\ll\gamma\ll 1$. In particular, $1/g'_s(n)\ll 1/M'$.

Let G be an n-vertex graph with $\alpha(G)\le g_s(n)$ with a 2-edge-colouring ϕ.

We apply Lemma 2.10 with $c=(g'_s(n)/n)^2$, to obtain a partition $V_1^*\cup V_2^*$ such that $\alpha(G_i[V_i^*])\le g'_s(n)$. Apply Theorem 2.5 with $G,V_1^*,V_2^*,\phi,\varepsilon,\varepsilon^{-1}$ and M' playing the roles of $G,U_1,U_2,\phi,\varepsilon,M$ and M' to obtain an ε-regular partition $V_0\cup V_1\cup\cdots\cup V_m$ with $\varepsilon^{-1}\le m\le M'$ which refines the partition $V_1^*\cup V_2^*$. Let $R:=R(\varepsilon,\gamma,\phi,(V_i)_{i\in[m]})$ be its reduced graph. Let the colouring ϕ^{ind} be as defined in the proof of Lemma 5.2. So if $\phi^{\mathrm{ind}}(i)=j$ for some $i\in V(R)$ and $j\in[2]$, it means the corresponding cluster V_i in G satisfies $\alpha(G_j[V_i])\le g'_s(n)$.

By Turán's Theorem, it suffices to show the following.

(R1) R is $K_{R(3,s)}$-free if ϕ is (K_3,K_{2s-1})-free;

(R2) R is $K_{R(3,s)+1}$-free if ϕ is (K_3,K_{2s})-free.

Indeed, it is easy to see that (R1) implies $e(G) \leq \frac{1}{2}\left(1 - \frac{1}{R(3,s)-1}\right) n^2 + 3\gamma n^2$ and (R2) implies $e(G) \leq \frac{1}{2}\left(1 - \frac{1}{R(3,s)}\right) n^2 + 3\gamma n^2$.

To show (R1) and (R2), without loss of generality, assume that $[t] \subseteq V(R)$ induces a maximum size clique in R. As the case $s = 2$ is covered in Theorem 1.1, we assume that $s \geq 3$.

Suppose that G is (K_3, K_{2s-1})-free (resp. (K_3, K_{2s})-free). Suppose that $\phi^{\text{ind}}(i) = 2$ for all $i \in [t]$, then by Lemma 2.8, $\phi^{\text{ind}}|_{[t]}$ is (K_3, K_s)-free, and so $t \leq R(3, s) - 1$ as desired. We may then assume that $\phi^{\text{ind}}(t) = 1$. Then $\phi^{\text{ind}}(it) = 2$ for all $i \in [t-1]$, as otherwise it is easy to see that one can embed K_3 in $G_1[V_i \cup V_t]$. Consequently, by Lemma 2.8, we have that $\phi^{\text{ind}}|_{[t-1]}$ is (K_3, K_{s-1})-free (resp. (K_3, K_s)-free). Hence, $t - 1 \leq R(3, s-1) - 1 \leq R(3, s) - 2$ (resp. $t - 1 \leq R(3, s) - 1$) as desired.

7.2. **Lower bound.** Let n be a sufficiently large number, and let $H(n)$ be an n-vertex K_3-free graph with independence number $O(\sqrt{n \log n})$. The celebrated result of Kim [17] shows the existence of such graphs.

7.2.1. *Lower bound for* $\mathrm{RT}(n, K_3, K_{2s-1}, g_s(n))$. Let $t = R(3, s) - 1$ and $\phi : \binom{[t]}{2} \to [2]$ be a (K_3, K_s)-free colouring. Let G be obtained from adding a copy of $H(n/t)$ to each partite set of $T_t(n)$. The following colouring witnesses G being (K_3, K_{2s-1})-free: colour all edges inside each partite set colour 2 and colour all crossing edges according to ϕ, i.e. for any $ij \in \binom{[t]}{2}$ and $h \in [2]$, all edges in $[X_i, X_j]$ are of colour h if $\phi(ij) = h$.

7.2.2. *Lower bound for* $\mathrm{RT}(n, K_3, K_{2s}, g_s(n))$. Let $t = R(3, s)$ and $\phi : \binom{[t-1]}{2} \to [2]$ be a (K_3, K_s)-free colouring. Let G be obtained from adding a copy of $H(n/t)$ to each partite set of $T_t(n)$. The following colouring witnesses G being (K_3, K_{2s})-free: colour all edges inside X_t colour 1, and edges inside X_i colour 2 for all $i \in [t-1]$; colour all crossing edges in $[X_1, \ldots, X_{t-1}]$ according to ϕ and colour all $[X_i, X_t]$-edges colour 2 for all $i \in [t-1]$.

8. Concluding remarks

8.1. **The value of $\varrho(K_3, K_6, \delta)$.** We conjecture that the following equality holds.

$$\varrho(K_3, K_6, \delta) = \frac{5}{12} + \frac{\delta}{2} + 2\delta^2.$$

The lower bound is given by the construction below, see Figure 2.

- Let $F_1 := F(\frac{n}{6}, d_1)$ and $F_2 := F(\frac{n}{6} - \frac{3\delta n}{2}, d_2)$ where $d_i \in [\delta n - o(n), \delta n]$. So $e(F_1) = \frac{\delta n^2}{12} \pm o(n^2)$ and $e(F_2) = \frac{\delta n^2}{12} - \frac{3\delta^2 n^2}{4} \pm o(n^2)$.
- Let $I = \{v_1, v_2, \ldots, v_{d_2}\}$ be an independent set of size d_2 in F_2. Let $I = I_1 \cup I_2$ be an equipartition of I. Let F be an $n/6$-vertex graph obtained from F_2 by
  - first adding 3 clone sets of I_1, say I_i with $i \in \{3, 4, 5\}$;
  - adding all $[I_i, I_{i+2}]$-edges for each $i \in [5]$ (addition modulo 5); and
  - adding an additional set of $\frac{3}{2}(\delta n - d_2)$ isolated vertices.

  Note that F is *not* triangle-free, and

$$e(F) = e(F_2) + \frac{3d_2}{2} \cdot d_2 + 5\left(\frac{d_2}{2}\right)^2 = \frac{\delta n^2}{12} + 2\delta^2 n^2 \pm o(n^2).$$

- Finally, let G be the graph obtained from $T_6(n)$, by putting a copy of F in X_6 and a copy of F_1 in X_i for each $i \in [5]$.

It is clear that G has the desired size and easy to check that the following 2-edge-colouring ϕ of G is (K_3, K_6)-free:

- let $\phi(X_i, X_{i+2}) = 1$ for each $i \in [5]$ (addition modulo 5);
- let $\phi(I_i, X_i \cup X_{i+1}) = 1$ for each $i \in [5]$ (addition modulo 5);

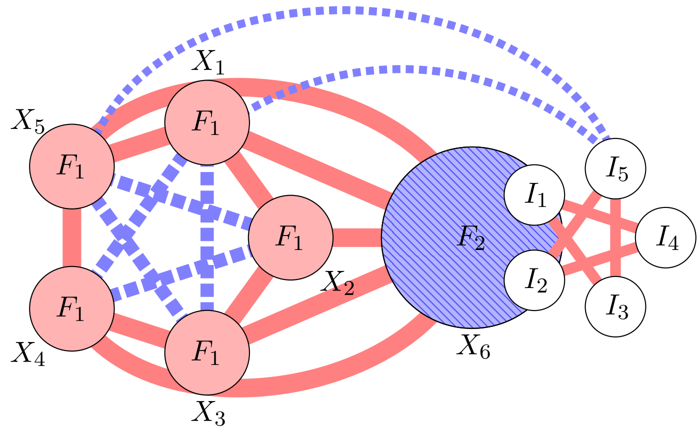

Figure 2: A graph with no blue (dotted) K_3 and no red K_6. All edges incident to $\bigcup_{i\in[5]} I_i$ and $\bigcup_{i\in[5]} X_i$ are omitted in the picture except blue edges between I_5 and $X_5 \cup X_1$.

- let $\phi(e) = 1$, for all $e \in E(G[X_6] \setminus G[\cup_{i\in[5]} I_i])$;
- all other edges are of colour 2.

8.2. **The value of $\varrho(K_3, K_{2s})$.** Recall that for triangle versus odd cliques, Erdős, Hajnal, Simonovits, Sós and Szemerédi [10] conjectured that $\varrho(K_3, K_{2s-1})$ is achieved by the $(R(3,s)-1)$-partite Turán graph, i.e. $\varrho(K_3, K_{2s-1}) = \frac{1}{2}\left(1 - \frac{1}{R(3,s)-1}\right)$. Base on Theorem 1.2, we put forward the following conjecture for triangle versus even cliques.

Conjecture 8.1. *For all $s \geq 2$, $\varrho(K_3, K_{2s}) = \frac{1}{2}\left(1 - \frac{1}{R(3,s)}\right)$.*

8.3. **Ramsey-Turán number with more than 2 colours.** We remark that the multicolour Ramsey-Turán number for *triangles* is related to a version of Ramsey number studied by Liu, Pikhurko and Sharifzadeh [19]. They introduced $r^*(K_{a_1}, \dots, K_{a_k})$ as the largest integer N such that there exists a colouring $\phi : \binom{[N]}{\leq 2} \to [k]$ with the following property:

($*$) for each $i \in [k]$, there is no edge-monochromatic K_{a_i} in colour i, and there is no edge incident to a vertex with the same colour, i.e. $\phi(ij) \neq \phi(i)$ for any $j \neq i$.

Note that when an n-vertex graph G is $(K_3, \dots, K_3)$-free with $\alpha(G) = o(n)$, then the colouring ϕ^{ind} on its reduced graph R satisfies ($*$), hence

$$\varrho(K_3, \dots, K_3) = \frac{1}{2}\left(1 - \frac{1}{r^*(K_3, \dots, K_3)}\right).$$

In particular, Theorems 1.8 and 1.9 in [19] imply $\varrho(K_3, K_3, K_3) = \frac{2}{5}$ and $\varrho(K_3, K_3, K_3, K_3) = \frac{15}{32}$.

In general Ramsey-Turán numbers for larger cliques are *not* determined by r^*, for example $\varrho(K_3, K_5) = \frac{2}{5} \neq \frac{1}{2}\left(1 - \frac{1}{r^*(K_3,K_5)}\right) = \frac{3}{8}$ and $\varrho(K_4, K_4) = \frac{11}{28} \neq \frac{1}{2}\left(1 - \frac{1}{r^*(K_4,K_4)}\right) = \frac{1}{3}$.

References

[1] J. Balogh, P. Hu, M. Simonovits, Phase transitions in the Ramsey-Turán theory, *J. Combin. Theory Ser. B*, 114, (2015), 148–169.

[2] J. Balogh, H. Liu, M. Sharifzadeh, On two problems in Ramsey-Turán theory, *SIAM J. Discrete Math.*, 31, (2017), 1848–1866.

[3] J. Balogh, A. McDowell, T. Molla, R. Mycroft, Triangle-tilings in graphs without large independent sets, *Combinatorics, Probability and Computing*, 27(4), (2018), 449–474.

[4] J. Balogh, T. Molla, M. Sharifzadeh, Triangle factors of graphs without large independent sets and of weighted graphs, with an Appendix by Christian Reiher and Mathias Schacht, *Random Structures and Algorithms*, 49, (2016), 669–693.

[5] T. Bohman, P. Keevash, Dynamic concentration of the triangle-free process, *The Seventh European Conference on Combinatorics, Graph Theory and Applications*, CRM Series, 16, (2013), 489–495.

[6] B. Bollobás, P. Erdős, On a Ramsey-Turán type problem, *J. Combin. Theory Ser. B*, 21, (1976), 166–168.

[7] S. Brandt, Triangle-free graphs whose independence number equals the degree, *Discrete Math.*, 310, (2010), 662–669.

[8] E. Davies, M. Jenssen, W. Perkins, B. Roberts, On the average size of independent sets in triangle-free graphs, *Proc. Amer. Math. Soc.*, to appear.

[9] P. Erdős, A. Hajnal, V. T. Sós, E. Szemerédi, More results on Ramsey-Turán type problems, *Combinatorica*, 3, (1983), 69–81.

[10] P. Erdős, A. Hajnal, M. Simonovits, V. T. Sós, E. Szemerédi, Turán-Ramsey theorems and simple asymptotically extremal structures, *Combinatorica*, 13, (1993), 31–56.

[11] P. Erdős, V.T. Sós, Some remarks on Ramseys and Turáns theorem, *Combinatorial theory and its applications, II* (Proc. Colloq., Balatonfured, 1969), North-Holland, Amsterdam, (1970), 395–404.

[12] P. Erdős, V. .T. Sós, On Turán-Ramsey type theorems II, *Studia Sci. Math. Hungar.*, 14, (1979), 27–36.

[13] J. Fox, P. Loh, Y. Zhao, The critical window for the classical Ramsey-Turán problem, *Combinatorica*, 35, (2015), 435–476.

[14] J. Fox, B. Sudakov, Dependent Random Choice, *Random Structures and Algorithms*, 38, (2011), 1–32.

[15] Z. Füredi, A proof of the stability of extremal graphs, Simonovits' stability from Szemerédi's regularity, *J. Combin. Theory Ser. B*, 115, (2015), 66–71.

[16] E. Győri, B. Keszegh, On the number of edge-disjoint triangles in K_4-free graphs, *Combinatorica*, to appear.

[17] J. H. Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$ *Random Structures and Algorithms*, 7, (1995), 173–207.

[18] J. Komlós, M. Simonovits, Szemerédi's regularity lemma and its applications to graph theory, In *Combinatorics, Paul Erdős is Eighty*, János Bolyai Math. Soc., Budapest, 2, (1996), 295–352.

[19] H. Liu, O. Pikhurko, M. Sharifzadeh, Edges not in any monochromatic copy of a fixed graph, *J. Combin. Theory Ser. B*, to appear.

[20] C. Lüders, C. Reiher, The Ramsey-Turán problem for cliques, *Israel Journal of Mathematics*, to appear.

[21] G. Fiz Pontiveros, S. Griffiths, R. Morris, The triangle-free process and the Ramsey number $R(3,k)$, *Mem. Amer. Math. Soc.*, to appear.

[22] J. Shearer, A note on the independence number of triangle-free graphs, *Discrete Math.*, 46, (1983), 83–87.

[23] M. Simonovits, V.T. Sós, Ramsey-Turán theory, *Discrete Math.*, 229, (2001), 293–340.

[24] B. Sudakov, A few remarks on RamseyTurán-type problems, *J. Combin. Theory Ser. B*, 88, (2003), 99–106.

[25] E. Szemerédi, On graphs containing no complete subgraph with 4 vertices, *Mat. Lapok*, 23, (1972), 113–116.

[26] P. Turán, Eine Extremalaufgabe aus der Graphentheorie, *Mat. Fiz. Lapok*, 48, (1941), 436–452.

Jaehoon Kim
School of Mathematics
University of Birmingham
Birmingham
UK

Younjin Kim
Institute of Mathematical Sciences
Ewha Womans University
Seoul
South Korea

Hong Liu
Mathematics Institute
University of Warwick
Coventry
UK

E-mail addresses: `j.kim.3@bham.ac.uk`, `younjinkim@ewha.ac.kr`, `h.liu.9@warwick.ac.uk`.